

# CORRELATION FUNCTIONS FOR RANDOM COMPLEX ZEROES: STRONG CLUSTERING AND LOCAL UNIVERSALITY

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**ABSTRACT.** We prove strong clustering of  $k$ -point correlation functions of zeroes of Gaussian Entire Functions. In the course of the proof, we also obtain universal local bounds for  $k$ -point functions of zeroes of arbitrary nondegenerate Gaussian analytic functions.

In the second part of the paper, we show that strong clustering yields the asymptotic normality of fluctuations of some linear statistics of zeroes of Gaussian Entire Functions, in particular, of the number of zeroes in measurable domains of large area. This complements our recent results from the paper “Fluctuations in random complex zeroes”.

## 1. INTRODUCTION

Consider the Gaussian entire function (G.E.F., for short)  $F(z) = \sum_{j \geq 0} \zeta_j \frac{z^j}{\sqrt{j!}}$ , where  $\zeta_j$  are standard independent Gaussian complex coefficients; that is, the density of the probability distribution of  $\zeta_j$  with respect to the Lebesgue measure on the complex plane is  $\frac{1}{\pi} e^{-|\zeta|^2}$ . A remarkable feature of the random zero set  $\mathcal{Z}_F = F^{-1}\{0\}$  is its distribution invariance with respect to the isometries of the complex plane [5, 10, 16]. It's easy to compute the covariance function of  $F$ :

$$\mathcal{E}\{F(z)\overline{F(w)}\} = \sum_{j \geq 0} \frac{z^j \overline{w}^j}{j!} = e^{z\overline{w}}.$$

Hence, after normalization, the covariance equals  $e^{z\overline{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2} = e^{i\operatorname{Im}(z\overline{w}) - \frac{1}{2}|z-w|^2}$ , which decays very fast when  $|z - w|$  grows. This hints at the almost independence of random zeroes at large distances. One of convenient ways to formalize the almost independence of a zero process at large distances is based on *clustering* of its  $k$ -point functions, cf. [14, § 4.4]. In this paper, we will develop this idea for zeroes of G.E.F.'s, and then will apply it to prove the asymptotic normality of fluctuations of some linear statistics of zeroes of Gaussian Entire Functions. We note that there is a very different approach to almost independence of random complex zeroes on large distances that proved to be useful in [11, 12, 13].

**1.1.  $k$ -point functions.** The  $k$ -point functions express correlations within  $k$ -point subsets of the point process. It is customary in statistical mechanics to describe point processes by the properties of their  $k$ -point correlation functions.

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The  $k$ -point function  $\rho = \rho_k$  of a random zero process  $\mathcal{Z}$  on  $\mathbb{C}$  is a symmetric function

$$\rho: \{Z = (z_1, \dots, z_k) \in \mathbb{C}^k : z_i \neq z_j, \text{ for } i \neq j\} \rightarrow \mathbb{R}_+$$

defined by the formula

$$\mathcal{E}\left\{\prod_{1 \leq i \leq k} \#(\mathcal{Z} \cap B_i)\right\} = \int_{B_1 \times \dots \times B_k} \rho(z_1, \dots, z_k) dA(z_1) \dots dA(z_k),$$

for any family of mutually disjoint bounded Borel sets  $B_1, \dots, B_k$  in  $\mathbb{C}$ . Here  $A$  is the Lebesgue measure on the complex plane. The  $k$ -point function of zeroes of a Gaussian analytic function  $f$  exists, provided that for any  $z_1, \dots, z_k \in \mathbb{C}$  with  $z_i \neq z_j$  for  $i \neq j$ , the random variables  $f(z_1), \dots, f(z_k)$  are linearly independent [5, Corollary 3.4.2]. It is not difficult to show (see the beginning of Section 2) that for G.E.F.'s, this condition holds for each  $k \in \mathbb{N}$ .

**1.2. Main results.** The first result treats the local behaviour of  $k$ -point functions. It appears that for a wide class of non-degenerate Gaussian analytic functions, the  $k$ -point functions of their zeroes exhibit universal local repulsion when some of the variables  $z_1, \dots, z_k$  approach each other.

Recall that a Gaussian analytic function (G.a.f., for short)  $f(z)$  in a plane domain  $G \subseteq \mathbb{C}$  is the sum

$$(1) \quad f(z) = \sum_n \zeta_n f_n(z)$$

of analytic functions  $f_n(z)$  such that

$$\sum_n |f_n(z)|^2 < \infty \quad \text{locally uniformly on } G,$$

with independent standard complex Gaussian coefficients  $\zeta_n$ .

We postpone until the beginning of Section 2 the technical definition of  $d$ -degeneracy, which we use in the assumptions of the next theorem. Here, we only mention that G.a.f.'s with "deterministic zeroes" (that is,  $f(z_0) = 0$  a.s., for some  $z_0 \in G$ ) are 1-degenerate. G.a.f.'s such that the random variables  $f(z_1), \dots, f(z_k)$  are linearly dependent for some  $z_1, \dots, z_k \in G$ , are  $k$ -degenerate, and G.a.f.'s for which the random variables  $f(z_1), f'(z_1), \dots, f(z_k), f'(z_k)$  are linearly dependent are  $2k$ -degenerate. We also mention that Gaussian Taylor series (either infinite, or finite)

$$f(z) = \sum_{n \geq 0} \zeta_n c_n z^n$$

are  $d$ -nondegenerate, provided that  $c_0, c_1, \dots, c_{d-1} \neq 0$ . In particular, the G.E.F. is  $d$ -nondegenerate for every positive integer  $d$ .

**Theorem 1.1.** *Let  $f$  be a  $2k$ -nondegenerate G.a.f. in a domain  $G$ , let  $\rho_f$  be a  $k$ -point function of zeroes of  $f$ , and let  $K \subset G$  be a compact set. Then there exists a positive constant  $C = C(k, f, K)$  such that, for any configuration of pairwise distinct points  $z_1, \dots, z_k \in K$ ,*

$$C^{-1} \prod_{i < j} |z_i - z_j|^2 \leq \rho_f(z_1, \dots, z_k) \leq C \prod_{i < j} |z_i - z_j|^2.$$

The next result is a clustering property of zeroes of G.E.F.'s. It says that if the variables in  $\mathbb{C}^k$  can be split into two groups located far from each other, then the function  $\rho_k$  almost equals the product of the corresponding factors. This property is another manifestation of almost independence of points of the process at large distances.

For a non-empty subset  $I = \{i_1, \dots, i_\ell\} \subset \{1, 2, \dots, k\}$ , we set  $Z_I = \{z_{i_1}, \dots, z_{i_\ell}\}$ . We denote by

$$d(Z_I, Z_J) = \inf_{i \in I, j \in J} |z_i - z_j|$$

the distance between the configurations  $Z_I$  and  $Z_J$ .

**Theorem 1.2.** *For each  $k \geq 2$ , there exist positive constants  $C_k$  and  $\Delta_k$  such that for each configuration  $Z$  of size  $k$  and each partition of the set of indices  $\{1, 2, \dots, k\}$  into two non-empty subsets  $I$  and  $J$  with  $d(Z_I, Z_J) \geq 2\Delta_k$ , one has*

$$(2) \quad 1 - \delta \leq \frac{\rho(Z)}{\rho(Z_I)\rho(Z_J)} \leq 1 + \delta \quad \text{with} \quad \delta = C_k e^{-\frac{1}{2}(d(Z_I, Z_J) - \Delta_k)^2}.$$

Combining Theorems 1.1 and 1.2, and taking into account the translation invariance of the random zero process  $Z_F$ , we obtain a uniform estimate for  $\rho_k$  valid in the whole  $\mathbb{C}^k$ :

**Theorem 1.3.** *For each  $k \geq 1$ , there exists a positive constant  $C_k$  such that for each configuration  $(z_1, \dots, z_k)$ ,*

$$C_k^{-1} \prod_{i < j} \ell(|z_i - z_j|) \leq \rho(z_1, \dots, z_k) \leq C_k \prod_{i < j} \ell(|z_i - z_j|),$$

where  $\ell(t) = \min(t^2, 1)$ .

To get Theorem 1.3, we use induction on  $k$ . For  $k = 1$  the result is obvious. Now, given an integer  $m \geq 2$ , suppose that Theorem 1.3 has been proven for  $k$ -point functions with  $k \leq m$ . To prove it for  $m+1$ -point functions, we consider two cases. First, suppose that the configuration of points  $\{z_1, \dots, z_{m+1}\}$  cannot be split into two groups lying at distance  $2\Delta_{m+1}$  or more from each other. Then the result follows from the local bounds in Theorem 1.1 and translation invariance. In the other case, the result follows from Theorem 1.2 and the inductive assumption.  $\square$

Note that Theorem 1.3 yields that the  $k$ -point functions  $\rho_k$  are uniformly bounded on  $\mathbb{C}^k$  by constants depending on  $k$ . This observation immediately yields the additive version of the clustering property:

**Theorem 1.4.** *For each  $k \geq 2$ , there exist positive constants  $C_k$  and  $\Delta_k$  such that for each configuration  $Z$  of size  $k$  and each partition of the set of indices  $\{1, 2, \dots, k\}$  into two non-empty subsets  $I$  and  $J$  with  $d(Z_I, Z_J) \geq 2\Delta_k$ , one has*

$$(3) \quad |\rho(Z) - \rho(Z_I)\rho(Z_J)| \leq C_k e^{-\frac{1}{2}(d(Z_I, Z_J) - \Delta_k)^2}.$$

The proofs of Theorems 1.1 and 1.2 start with the classical Kac-Rice-Hammersley formula [5, Chapter 3]:

$$(4) \quad \rho_f(z_1, \dots, z_k) = \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 \mathcal{D}_f(\eta'; z_1, \dots, z_k) dm(\eta_1) \dots dm(\eta_k),$$

where  $\mathcal{D}_f(\cdot; z_1, \dots, z_k)$  is the density of the joint probability distribution of the random variables

$$(5) \quad f(z_1), f'(z_1), \dots, f(z_k), f'(z_k),$$

and  $\eta' = (0, \eta_1, \dots, 0, \eta_k)^T$  is a vector in  $\mathbb{C}^{2k}$ . Since the random variables (5) are complex Gaussian, one can rewrite the right-hand side of (4) in a more explicit form

$$(6) \quad \rho_f(z_1, \dots, z_k) = \frac{1}{\pi^{2k} \det \Gamma_f} \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} \langle \Gamma_f^{-1} \eta', \eta' \rangle} dm(\eta_1) \dots dm(\eta_k),$$

where  $\Gamma_f = \Gamma_f(z_1, \dots, z_k)$  is the covariance matrix of the random variables (16). We consider the linear functionals

$$Lf = \sum_{j=1}^k [\alpha_j f(z_j) + \beta_j f'(z_j)] = \frac{1}{2\pi i} \int_{\gamma} f(z) r^L(z) dz,$$

where

$$r^L(z) = \sum_{j=1}^k \left[ \frac{\alpha_j}{z - z_j} + \frac{\beta_j}{(z - z_j)^2} \right],$$

and  $\gamma \subset K$  is a smooth contour that bounds a domain  $G' \subset K$  that contains the points  $z_1, \dots, z_k$ . Then we observe that for every vector  $\delta = (\alpha_1, \beta_1, \dots, \alpha_k, \beta_k)^T$  in  $\mathbb{C}^{2k}$ , we have

$$\langle \Gamma_f \delta, \delta \rangle = \mathcal{E} |Lf|^2.$$

This observation allows us to estimate the matrix  $\Gamma_f^{-1}$ , and hence the integral on the right-hand side of (6), using some simple tools from the theory of analytic functions of one complex variable, and thus avoiding formidable expressions for the Gaussian integrals on the right-hand side of (6) that involve quotients of large determinants and permanents.

We note that in [2], Bleher, Shiffman, and Zelditch estimated these large determinants and permanents, and proved that if the points  $z_i$  are well separated from each other, i.e.,  $\min_{i \neq j} |z_i - z_j| \geq \eta > 0$ , then some estimate similar to (3) holds with a factor  $C(k, \eta)$  on the right-hand side. Unfortunately, in this form their result is difficult to apply. For instance, it does not yield boundedness of the  $k$ -point functions on the whole  $\mathbb{C}^k$ , and we could not use it for the proof of the asymptotic normality of some linear statistics, see Theorem 1.5 below.

**1.3. Clustering of  $k$ -point functions and asymptotic normality of linear statistics.** Various ways to derive the asymptotic normality of fluctuations of the number of random points in large volumes from clustering of all  $k$ -point functions are known in statistical mechanics. Our next result is a variation on this theme. It pertains to arbitrary point processes  $\mathcal{Z}$  on the plane with clustering  $k$ -point functions.

Let  $h$  be a non-zero bounded measurable function with compact support. For the scaled linear statistics  $n(R; h) = \sum_{a \in \mathcal{Z}} h(\frac{a}{R})$  of the point process  $\mathcal{Z}$ , we set

$$\bar{n}(R; h) = n(R; h) - \mathcal{E}\{n(R; h)\}, \quad \sigma(R; h)^2 = \mathcal{E}\{\bar{n}(R; h)^2\},$$

$$\text{and} \quad n^*(R; h) = \frac{\bar{n}(R; h)}{\sigma(R; h)}.$$

In what follows, we consider the functions  $h$  with  $\sigma(R; h)$  growing as a positive power of  $R$ , i.e., satisfying

$$(7) \quad \sigma(R; h) \geq R^\delta$$

for some  $\delta > 0$  and for all sufficiently big  $R$ . Later, we will see that this holds when  $h$  is an indicator function of an arbitrary bounded measurable set of positive area (Lemma 1.6 below).

We call the function  $\varphi: (0, \infty) \rightarrow (0, \infty)$  *fast decreasing* if it decreases and for each positive  $m$ ,  $\lim_{x \rightarrow \infty} x^m \varphi(x) = 0$ . We say that the  $k$ -point functions  $\rho$  of a point process  $\mathcal{Z}$  are *clustering* if there exists a fast decreasing function  $\varphi$  such that for each  $k \geq 2$  and for each partition of the set of indices  $\{1, 2, \dots, k\}$  into non-empty disjoint subsets  $I$  and  $J$ , one has

$$(8) \quad |\rho(Z) - \rho(Z_I)\rho(Z_J)| \leq C_k \varphi(c_k d(Z_I, Z_J)).$$

We say that the  $k$ -point functions are bounded, if  $\sup_{\mathbb{C}^k} \rho_k < \infty$ .

**Theorem 1.5.** *Suppose that the  $k$ -point correlation functions  $\rho_k$  of a point process  $\mathcal{Z}$  are clustering and bounded. Then for each bounded measurable compactly supported function  $h$  with  $\sigma(R; h)$  growing as a positive power of  $R$ , the normalized linear statistics  $n^*(R; h)$  converge in distribution to the standard Gaussian law as  $R \rightarrow \infty$ .*

The proof of this theorem is based on estimates for cumulants that follow from clustering. A similar approach was used by Malyshev in [7], and by Martin and Yalçın in [9].

We note that there is a counterpart of Theorem 1.5 for arbitrary determinantal point processes, namely, a theorem of Soshnikov. In [17], he proved that for arbitrary determinantal point processes, the fluctuations of linear statistics associated with compactly supported bounded positive functions  $h$  are asymptotically normal if the variance grows at least as a positive power of the expectation. His proof is based on peculiar combinatorial identities for the cumulants of linear statistics that are a special feature of determinantal point processes, and is quite different from the one of Theorem 1.5.

**1.4. Lower bounds for the variance of linear statistics.** In order to apply Theorem 1.5, we need to know that the variance of the linear statistics  $n(R; h)$  grows as a positive power of  $R$ . An example of such statistics is the number of points of the process  $\mathcal{Z}$  in a bounded set  $E$  of positive measure dilated  $R$  times.

We assume that  $\mathcal{Z}$  is a *translation-invariant* point process on the plane with one- and two-point correlation functions and that the mean number of points of the process  $\mathcal{Z}$  per unit area equals 1; that is,  $\rho_1 \equiv 1$  on  $\mathbb{R}^2$ . We also assume that the 2-point function of the point process  $\mathcal{Z}$  satisfies

$$(9) \quad \rho(z_1, z_2) = r(z_1 - z_2) \quad \text{with} \quad r - 1 \in L^1(\mathbb{R}^2)$$

which is a weak form of clustering. For a set  $E$ , we denote by  $\mathbb{1}_E$  its indicator function.

**Lemma 1.6.** *Let  $\mathcal{Z}$  be a translation-invariant random point process on  $\mathbb{R}^2$  with the 2-point function satisfying (9). Then there exists a numerical constant  $c > 0$  such that for each bounded measurable set  $E \subset \mathbb{R}^2$  of positive area  $A(E)$ , one has*

$$\sigma(R, \mathbb{1}_E)^2 \geq c \min \left\{ A(E)R^2, \sqrt{A(E)}R \right\}, \quad 0 < R < \infty.$$

Note that we do not impose any smoothness assumption on the boundary of the set  $E$ . The proof of Lemma 1.6 is based on a simple observation:

**Lemma 1.7.** *Suppose  $\mathcal{Z}$  is a translation-invariant random point process on  $\mathbb{R}^2$  with the 2-point function satisfying (9). Then there exists a numerical constant  $C > 0$  such that for every function  $h \in (L^1 \cap L^2)(\mathbb{R}^2)$  and every  $R > 0$ ,*

$$(10) \quad \sigma(R, h)^2 \geq \frac{R^2}{2} \int_{|\xi| \geq CR} |\widehat{h}(\xi)|^2 dA(\xi),$$

where

$$\widehat{h}(\xi) = \int_{\mathbb{R}^2} h(x) e^{-2\pi i x \cdot \xi} dA(x)$$

is the Fourier transform of  $h$ .

Estimate (10) reduces lower bounds for  $\sigma(R, h)$  to estimates for the tail of the integral on the right-hand side of (10).

**1.5. Related results on asymptotic normality of fluctuations of linear statistics of random complex zeroes.** Theorem 1.5 complements our recent results [11] on fluctuations in the random complex zeroes  $\mathcal{Z}_F$  obtained by a completely different technique. Therein, we showed that if  $\mathcal{Z}_F$  is a zero set of a G.E.F.  $F(z)$ , then the following hold:

- (i) If  $h$  is a  $C_0^\alpha$ -function with  $\alpha > 1$ , then the fluctuations of  $n(R, h)$  are asymptotically normal.<sup>1</sup>
- (ii) If  $h$  is a  $C_0^\alpha$  function with  $0 < \alpha \leq 1$ , and if for some  $\varepsilon > 0$ , we have  $\sigma(R, h) \geq R^{-\alpha+\varepsilon}$ , then the fluctuations of  $n(R, h)$  are asymptotically normal as well.
- (iii) For each  $0 < \alpha < 1$ , there are  $C_0^\alpha$ -functions  $h$  with abnormal fluctuations of linear statistics  $n(R, h)$ .

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## 2. LOCAL UNIVERSALITY OF $k$ -POINT FUNCTIONS. PROOF OF THEOREM 1.1

Let  $G \subseteq \mathbb{C}$  be a plane domain. We fix a compact set  $K$  and a bounded domain  $G_1$  with a smooth boundary  $\gamma = \partial G_1$  such that  $K \subset G_1$  and  $\overline{G_1} \subset G$ . We consider linear functionals

$$(11) \quad Lg = \sum_{j=1}^k a_j g^{(m_j)}(z_j)$$

with  $z_1, \dots, z_k \in K$ , acting on functions  $g$  analytic in  $G$ . Then for every function  $g$  analytic in  $G$  and for every functional  $L$  as above, we have

$$(12) \quad Lg = \int_{\gamma} g(z) r^L(z) dz \quad \text{with} \quad r^L(z) = \frac{1}{2\pi i} \sum_{j=1}^k \frac{a_j m_j!}{(z - z_j)^{m_j+1}}.$$

Note that  $r^L$  is a rational function vanishing at infinity. We put

$$\text{rank}(L) \stackrel{\text{def}}{=} \deg(r^L).$$

Then we say that a G.a.f.  $f(z)$  in  $G$  is  $d$ -degenerate, if there exists a functional  $L$  of rank at most  $d$  such that

$$(13) \quad Lf = 0 \quad \text{almost surely.}$$

Otherwise, we say that  $f$  is  $d$ -nondegenerate. For instance, G.a.f.'s with “deterministic zeroes” (that is,  $f(z_0) = 0$  a.s., for some  $z_0 \in G$ ) are 1-degenerate. G.a.f.'s such that the random variables  $f(z_1), \dots, f(z_k)$  are linearly dependent are  $k$ -degenerate, and G.a.f.'s for which the random variables  $f(z_1), f'(z_1), \dots, f(z_k), f'(z_k)$  are linearly dependent are  $2k$ -degenerate.

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<sup>1</sup>This result was preceded by yet different proofs in [16, 18] of the asymptotic normality in the case of  $C_0^2$ -functions  $h$ .

Observe that the G.a.f. (1) satisfies condition (13) if and only if  $Lf_n = 0$  for every  $n$ . So Gaussian Taylor series (infinite, or finite)

$$f(z) = \sum_{n \geq 0} \zeta_n c_n z^n$$

are  $d$ -nondegenerate provided that  $c_0, c_1, \dots, c_{d-1} \neq 0$ . Indeed, suppose that (13) holds. Then

$$Lz^n = \int_{\gamma} z^n r^L(z) dz = 0, \quad \text{for } n = 0, 1, \dots, d-1.$$

Choose  $R$  so large that  $G \subset \{|z| < R\}$ . Then

$$\int_{\{|z|=R\}} z^n r^L(z) dz = \int_{\gamma} z^n r^L(z) dz = 0, \quad \text{for } n = 0, 1, \dots, d-1,$$

whence,

$$r^L(z) = O(z^{-(d+1)}), \quad z \rightarrow \infty.$$

That is, the degree of  $r^L$  is not less than  $d+1$ , which means that  $f(z)$  is  $d$ -nondegenerate, as we claimed. In particular, we see that the G.E.F. is  $d$ -nondegenerate for every non-negative integer  $d$ .

**Claim 2.1.** *Let  $f$  be a  $d$ -nondegenerate G.a.f., and let  $K \subset G$  be a compact set. Then there exists positive constants  $c(d, f, K)$  and  $C(f, K)$  such that for each functional  $L$  of rank at most  $d$  with  $z_1, \dots, z_k \in K$ , we have*

$$(14) \quad c \max_{\gamma} |r^L|^2 \leq \mathcal{E}|Lf|^2 \leq C \max_{\gamma} |r^L|^2.$$

*Proof of the upper bound in (14):*

$$\begin{aligned} \mathcal{E}|Lf|^2 &= \mathcal{E} \left| \int_{\gamma} f r^L dz \right|^2 \\ &\leq \left( \text{length}(\gamma) \cdot \max_{\gamma} |r^L| \right)^2 \cdot \int_{\gamma} \mathcal{E}|f|^2 |dz| = C(f, \gamma) \left( \max_{\gamma} |r^L| \right)^2. \quad \square \end{aligned}$$

The proof of the lower bound uses a simple compactness argument. Denote by  $R_d(K)$  the set of rational functions of degree at most  $d$  vanishing at  $\infty$  and having all poles in  $K$ .

**Claim 2.2.** *Each sequence of rational functions  $\{r_m\} \subset R_d(K)$  with  $\max_{\gamma} |r_m| \leq 1$  has a subsequence that converges uniformly on  $\gamma$  to a function  $r \in R_d(K)$ .*

*Proof of Claim 2.2:* We have  $r_m = \frac{p_m}{q_m}$ , where  $p_m$  and  $q_m$  are polynomials of degree  $\leq d-1$  and  $\leq d$  correspondingly, and  $q_m(z) = \prod_{i=1}^d (z - z_i(m))$  with  $z_1(m), \dots, z_d(m) \in K$  for each  $m$ . Choosing a subsequence, we may assume that  $q_m$  converge uniformly on  $\gamma$  to a polynomial  $q$  of the same form. Then  $\max_{\gamma} |p_m| \leq C$  for each  $m$ . Since the degrees of the polynomials  $p_m$  are uniformly bounded, we can choose a subsequence that converges uniformly on  $\gamma$  to a polynomial of degree  $\leq d-1$ .  $\square$

*Proof of the lower bound in (14):* Suppose that there exists a sequence of functionals  $L_m$  of the form (11) of rank  $d$  or less, with  $\max_{\gamma} |r^{L_m}| = 1$  and with  $\mathcal{E}|L_m f|^2 \rightarrow 0$ . By (12) and Claim 2.2, we can choose a subsequence of this sequence such that  $r^{L_m}$



converge uniformly on  $\gamma$  to some nonzero  $r \in R_d(K)$  (to simplify our notation, we omit subindices). Then  $r = r^L$  for some functional  $L$  of the same form (11), and

$$\lim_{m \rightarrow \infty} \mathcal{E}|L_m f - Lf|^2 \leq (\text{length}(\gamma))^2 \cdot \int_{\gamma} \mathcal{E}|f|^2 |dz| \cdot \lim_{m \rightarrow \infty} \max_{\gamma} |r^{L_m} - r^L|^2 = 0.$$

Hence,  $Lf = 0$  almost surely, which contradicts the  $d$ -nondegeneracy of  $f$ .  $\square$

To estimate the  $k$ -point function, we use a classical formula that goes back to Kac, Rice, and Hammersley, see [2] and [5, Chapter 3]:

$$(15) \quad \rho_f(z_1, \dots, z_k) = \frac{1}{\pi^{2k} \det \Gamma_f} \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} \langle \Gamma_f^{-1} \eta', \eta' \rangle} dA(\eta_1) \dots dA(\eta_k),$$

where  $\Gamma_f = \Gamma_f(z_1, \dots, z_k)$  is the covariance matrix of the random variables

$$(16) \quad f(z_1), f'(z_1), \dots, f(z_k), f'(z_k),$$

and  $\eta' = (0, \eta_1, \dots, 0, \eta_k)^T$  is a vector in  $\mathbb{C}^{2k}$ . Here, we assume that the random variables (16) are linearly independent; later, we will impose on the G.a.f.  $f(z)$  a somewhat stronger restriction of non-degeneracy. The Kac-Rice-Hammersley formula (15) allows us to reduce estimates for the  $k$ -point function  $\rho_f$  to estimates for the covariance matrix  $\Gamma_f$ .

We start with a simple observation. Consider “special” linear functionals of rank  $2k$ :

$$(17) \quad Lf = \sum_{j=1}^k [\alpha_j f(z_j) + \beta_j f'(z_j)].$$

**Claim 2.3.** *For every vector  $\delta = (\alpha_1, \beta_1, \dots, \alpha_k, \beta_k)^T$  in  $\mathbb{C}^{2k}$ , we have  $\langle \Gamma_f \delta, \delta \rangle = \mathcal{E}|Lf|^2$ .*

*Proof:* by straightforward inspection.  $\square$

Introduce the Gaussian polynomial

$$f_{2k-1}(z) = \sum_{n=0}^{2k-1} \zeta_n z^n.$$

**Claim 2.4.** *Let  $f$  be a  $2k$ -nondegenerate G.a.f. and let  $K \subset G$  be a compact set. Then there exists a positive constant  $C(k, f, K)$  such that for each configuration  $z_1, \dots, z_k \in K$ ,*

$$(18) \quad C^{-1} \Gamma_{f_{2k-1}}(z_1, \dots, z_k) \leq \Gamma_f(z_1, \dots, z_k) \leq C \Gamma_{f_{2k-1}}(z_1, \dots, z_k),$$

where the inequalities are understood in the operator sense.

*Proof:* this is a straightforward consequence of Claims 2.3 and 2.1.  $\square$

Now, using the Kac-Rice-Hammersley formula (15), we readily get equivalence of the  $k$ -point functions  $\rho_f$  and  $\rho_{f_{2k-1}}$ :

**Claim 2.5.** *Let  $f$  be a  $2k$ -nondegenerate G.a.f. and let  $K \subset G$  be a compact set. Then there exists a positive constant  $C(k, f, K)$  such that for each configuration  $z_1, \dots, z_k \in K$ ,*

$$(19) \quad C^{-1} \rho_{f_{2k-1}}(z_1, \dots, z_k) \leq \rho_f(z_1, \dots, z_k) \leq C \rho_{f_{2k-1}}(z_1, \dots, z_k).$$

*Proof:* First, note that Claim 2.4 yields

$$(20) \quad C^{-2k} \det \Gamma_{f_{2k-1}} \leq \det \Gamma_f \leq C^{2k} \det \Gamma_{f_{2k-1}}.$$



Then, plugging estimates (18) and (20) into (15), we get

$$\begin{aligned} \rho_f(z_1, \dots, z_k) &\leq \frac{C^{2k}}{\pi^{2k} \det \Gamma_{f_{2k-1}}} \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} C^{-1} \langle \Gamma_{f_{2k-1}}^{-1} \eta', \eta' \rangle} dA(\eta_1) \dots dA(\eta_k) \\ &= \frac{C^{2k}}{\pi^{2k} \det \Gamma_{f_{2k-1}}} \cdot C^{2k} \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} \langle \Gamma_{f_{2k-1}}^{-1} \eta', \eta' \rangle} dA(\eta_1) \dots dA(\eta_k) \\ &= C^{4k} \rho_{f_{2k-1}}(z_1, \dots, z_k). \end{aligned}$$

Similarly, we get the lower bound in (19).  $\square$

We see that it suffices to estimate the  $k$ -point function only for one special Gaussian polynomial  $f_{2k-1}$  of degree  $2k - 1$ . First we consider the probability density  $\mathbf{p}_{f_{2k-1}}(z_1, \dots, z_{2k-1})$  of the joint distribution of all zeroes of  $f_{2k-1}$ . This means that  $\mathbf{p}_{f_{2k-1}}$  is a symmetric function such that, for any symmetric function  $S$  of  $2k - 1$  complex variables, the expected value of  $S(z_1, \dots, z_{2k-1})$  equals

$$\int_{\mathbb{C}^k} S(z_1, \dots, z_{2k-1}) \mathbf{p}_{f_{2k-1}}(z_1, \dots, z_{2k-1}) dA(z_1) \dots dA(z_{2k-1}).$$

The density  $\mathbf{p}_{f_{2k-1}}$  can be computed using a classical formula for the Jacobian of the transformation of zeroes of the polynomial into its Taylor coefficients [3, 4]:

$$\mathbf{p}_{f_{2k-1}}(z_1, \dots, z_{2k-1}) = C_k \prod_{1 \leq i < j \leq 2k-1} |z_i - z_j|^2 \cdot \left( \sum_{0 \leq j \leq 2k-1} |\sigma_j|^2 \right)^{-2k},$$

where  $\sigma_j$ 's are the coefficients of the polynomial

$$\prod_{1 \leq i \leq 2k-1} (z - z_i) = \sum_{0 \leq j \leq 2k-1} \sigma_j z^j \quad (\text{here, } \sigma_{2k-1} = 1).$$

That is,

$$\mathbf{p}_{f_{2k-1}}(z_1, \dots, z_{2k-1}) = \prod_{1 \leq i < j \leq k} |z_i - z_j|^2 \cdot H(z_1, \dots, z_{2k-1}),$$

where  $H$  is a non-negative continuous function.

**Claim 2.6.** *There exists a positive constant  $C(k, K)$  such that for all  $z_1, \dots, z_k \in K$ , and all  $z_{k+1}, \dots, z_{2k-1} \in \mathbb{C}$ , we have*

$$H(z_1, \dots, z_{2k-1}) \leq C(k, K) \prod_{i=k+1}^{2k-1} (1 + |z_i|)^{-4}.$$

*Proof of Claim 2.6:* Let

$$P(z) = \prod_{i=1}^{2k-1} (z - z_i).$$

For each  $z \in \mathbb{T}$ , we have

$$|P(z)| \leq \sum_{j=0}^{2k-1} |\sigma_j|, \quad \text{whence} \quad \sum_{j=0}^{2k-1} |\sigma_j|^2 \geq \frac{1}{2k} |P(z)|^2.$$

An elementary geometry shows that there exists a point  $z \in \mathbb{T}$  so that  $|z - z_i| \geq c_k$  for  $i = 1, \dots, 2k - 1$ , whence,  $|z - z_i| \geq c_k(1 + |z_i|)$  for  $i = 1, \dots, 2k - 1$ , and

$$|P(z)|^2 \geq c_k \prod_{i=1}^{2k-1} (1 + |z_i|)^2 \geq c_k \prod_{i=k+1}^{2k-1} (1 + |z_i|)^2.$$

Therefore,

$$\sum_{j=0}^{2k-1} |\sigma_j|^2 \geq c_k \prod_{i=k+1}^{2k-1} (1 + |z_i|)^2,$$

and

$$\mathfrak{p}_{f_{2k-1}}(z_1, \dots, z_{2k-1}) \leq C_k \prod_{1 \leq i < j \leq 2k-1} |z_i - z_j|^2 \cdot \prod_{i=k+1}^{2k-1} (1 + |z_i|)^{-4k}.$$

At last, using that  $|z_i - z_j| \leq (1 + |z_i|)(1 + |z_j|)$ , we get

$$\begin{aligned} H(z_1, \dots, z_{2k-1}) &\leq C(k, K) \prod_{i=k+1}^{2k-1} (1 + |z_i|)^{2(2k-2)} \prod_{i=k+1}^{2k-1} (1 + |z_i|)^{-4k} \\ &= C(k, K) \prod_{i=k+1}^{2k-1} (1 + |z_i|)^{-4} \end{aligned}$$

completing the proof of the claim.  $\square$

Now, we readily finish the proof of the local universality theorem. The  $k$ -point function  $\rho_{f_{2k-1}}(z_1, \dots, z_k)$  can be obtained from the probability density function  $\mathfrak{p}_{f_{2k-1}}$  by integrating out the extra variables

$$\begin{aligned} \rho_{f_{2k-1}}(z_1, \dots, z_k) &= C_k \int_{\mathbb{C}^{k-1}} \mathfrak{p}_{f_{2k-1}}(z_1, \dots, z_{2k-1}) dA(z_{k+1}) \dots dA(z_{2k-1}) \\ &= C_k \prod_{1 \leq i < j \leq k} |z_i - z_j|^2 \cdot \int_{\mathbb{C}^{k-1}} H(z_1, \dots, z_{2k-1}) dA(z_{k+1}) \dots dA(z_{2k-1}). \end{aligned}$$

Due to Claim 2.6, the integral on the right-hand side converges uniformly with respect to the variables  $z_1, \dots, z_k \in K$ . Hence, it is a continuous function in  $z_1, \dots, z_k$ . Given  $z_1, \dots, z_k$ , the function  $H$  (as a function of variables  $z_{k+1}, \dots, z_{2k-1}$ ) vanishes only on a set of zero Lebesgue measure in  $\mathbb{C}^{k-1}$ , hence, the integral is a positive function of  $z_1, \dots, z_k$ . This completes the proof of Theorem 1.1.  $\square$

### 3. CLUSTERING OF $k$ -POINT FUNCTIONS. PROOF OF THEOREM 1.2

The proof again employs the linear functionals  $L$  introduced in (17) and Kac-Rice-Hammersley's formula (15). By  $\rho\mathbb{T}$  we denote the circumference  $\{|z| = \rho\}$ .

Let  $F$  be a G.E.F.. For  $w \in \mathbb{C}$ , denote by  $T_w$  the projective shift

$$T_w F(z) = F(w + z) e^{-z\overline{w}} e^{-\frac{1}{2}|w|^2}.$$

Comparing the covariances  $\mathcal{E} \left\{ F(z_1) \overline{F(z_2)} \right\}$  and  $\mathcal{E} \left\{ T_w F(z_1) \overline{T_w F(z_2)} \right\}$ , one readily checks that for every  $w \in \mathbb{C}$ ,  $T_w F$  is also a G.E.F.. Our first aim is to estimate the covariance  $\mathcal{E} \left\{ L_1(T_{w_1} F) \overline{L_2(T_{w_2} F)} \right\}$ , where  $L_1, L_2$  are two linear functionals of the kind (17), and  $w_1, w_2 \in \mathbb{C}$ ,  $w_1 \neq w_2$ .

**Claim 3.1.** *Let  $k \geq 1$ ,  $\rho > 0$ ,  $|w_1 - w_2| \geq 4\rho$ , and let the points  $z_j$  participating in the definition of the functional  $L$  satisfy  $|z_j| \leq \rho$  ( $j = 1, \dots, k$ ). Then*

$$(21) \quad \left| \mathcal{E} \left\{ L_1(T_{w_1}F) \overline{L_2(T_{w_2}F)} \right\} \right| \leq C(\rho, k) e^{-\frac{1}{2}(|w_1 - w_2| - 4\rho)^2} (\mathcal{E}|L_1 T_{w_1} F|^2 + \mathcal{E}|L_2 T_{w_2} F|^2).$$

Note that, since  $T_w F$  is equidistributed with  $F$  for all  $w \in \mathbb{C}$ , we can also rewrite the sum on the right-hand side as  $\mathcal{E}|L_1 F|^2 + \mathcal{E}|L_2 F|^2$ .

*Proof of Claim 3.1:* Using the representation (12) with  $\gamma = 2\rho\mathbb{T}$ , we can write

$$\begin{aligned} & \left| \mathcal{E} \left\{ L_1(T_{w_1}F) \overline{L_2(T_{w_2}F)} \right\} \right| \\ &= \left| \iint_{2\rho\mathbb{T} \times 2\rho\mathbb{T}} r^{L_1}(z_1) \overline{r^{L_2}(z_2)} \mathcal{E} \left\{ T_{w_1}F(z_1) \overline{T_{w_2}F(z_2)} \right\} dz_1 dz_2 \right| \\ &\leq \frac{1}{2} (2\pi\rho)^2 \cdot \left( \max_{2\rho\mathbb{T}} |r^{L_1}|^2 + \max_{2\rho\mathbb{T}} |r^{L_2}|^2 \right) \cdot \sup_{z_1, z_2 \in 2\rho\mathbb{T}} \left| \mathcal{E} \left\{ T_{w_1}F(z_1) \overline{T_{w_2}F(z_2)} \right\} \right|. \end{aligned}$$

Since for any  $w_1, w_2, \lambda_1, \lambda_2 \in \mathbb{C}$ , we have

$$\left| \mathcal{E} \left\{ T_{w_1}F(\lambda_1) \overline{T_{w_2}F(\lambda_2)} \right\} \right| = e^{\frac{1}{2}|\lambda_1|^2 + \frac{1}{2}|\lambda_2|^2 - \frac{1}{2}|(w_1 + \lambda_1) - (w_2 + \lambda_2)|^2},$$

the last supremum does not exceed  $e^{4\rho^2} e^{-\frac{1}{2}(|w_1 - w_2| - 4\rho)^2}$ , provided that  $|w_1 - w_2| \geq 4\rho$ . Taking into account that  $|r^{L_i}|^2 \leq C(k, \rho) \mathcal{E}|L_i F|^2$  everywhere on  $2\rho\mathbb{T}$  (Claim 2.1), we get (21).  $\square$

In what follows, we also need the following simple geometric claim.

**Claim 3.2.** *Suppose that  $z_1, \dots, z_k \in \mathbb{C}$ , and  $\underline{d}: (0, +\infty) \rightarrow (0, +\infty)$  is any increasing function. Then there exists  $\rho \geq 1$  and a covering of points  $z_j$  ( $j = 1, \dots, k$ ) by disks  $D(w_n, \rho)$  ( $n = 1, \dots, N$ ) such that  $N \leq k$  and  $|w_{n'} - w_{n''}| \geq \underline{d}(\rho)$  for all  $n' \neq n''$ . Moreover,  $\rho$  is bounded from above by some constant depending on  $\underline{d}$  and  $k$  only.*

*Proof of Claim 3.2:* Take  $\rho_1 = 1$  and consider the covering by the disks  $D(w_n, \rho_1)$  with  $N = k$ ,  $w_n = z_n$  for all  $n$ . If all pairwise distances  $|w_{n'} - w_{n''}| \geq \underline{d}(\rho_1)$ , we are done. If not, replace two centers  $w_{n'}$  and  $w_{n''}$  satisfying  $|w_{n'} - w_{n''}| < \underline{d}(\rho_1)$  by one center  $\frac{1}{2}(w_{n'} + w_{n''})$  and increase all radii to  $\rho_2 = \rho_1 + \frac{1}{2}\underline{d}(\rho_1)$ . If in the resulting configuration all pairwise distances between centers are at least  $\underline{d}(\rho_2)$ , we are done. Otherwise, again, replace two close centers by one and increase the radii to  $\rho_3 = \rho_2 + \frac{1}{2}\underline{d}(\rho_2)$ , and so on.

The process will stop after at most  $k - 1$  steps, and the final radius will be bounded by the  $k$ -th term in the recursive sequence given by  $\rho_1 = 1$ , and  $\rho_{j+1} = \rho_j + \frac{1}{2}\underline{d}(\rho_j)$ .  $\square$

**Claim 3.3.** *There exists a constant  $\Delta = \Delta(k)$  with the following property. Let  $z_1, \dots, z_k \in \mathbb{C}$ . Suppose that the set of indices  $\{1, 2, \dots, k\}$  is partitioned into two non-empty subsets  $I$  and  $J$  so that*

$$d = d(Z_I, Z_J) = \inf \{|z_i - z_j| : i \in I, j \in J\} \geq \Delta.$$

*Then for any two linear functionals*

$$L^I f = \sum_{i \in I} [\alpha_i f(z_i) + \beta_i f'(z_i)], \quad L^J f = \sum_{j \in J} [\alpha_j f(z_j) + \beta_j f'(z_j)],$$

one has

$$(22) \quad \left| \mathcal{E} \left\{ (L^I F) (\overline{L^J F}) \right\} \right| \leq C(k) e^{-\frac{1}{2}(d-\Delta)^2} \{ \mathcal{E} |L^I F|^2 + \mathcal{E} |L^J F|^2 \}.$$

*Proof of Claim 3.3:* We put

$$\underline{d}(\rho) = 4\rho + \sqrt{2 \log(10k C(\rho, k))}$$

where the constant  $C(\rho, k)$  is the same as in (21), and apply the construction of Claim 3.2 to the points  $z_j$ . We get a number  $\rho \leq \rho(k)$  with  $\rho(k) \geq 1$  independent of  $z_1, \dots, z_k$ , and a sequence of disjoint disks  $D(w_n, \rho)$  ( $n = 1, \dots, N$ ,  $N \leq k$ ) such that each point  $z_j$  lies in at least one of these disks. Then we take  $\Delta = 6\rho(k)$ , and notice that no disk can contain two points  $z_i, z_j$  with  $i \in I$ , and  $j \in J$ . Thus, the set of indices  $\{1, 2, \dots, N\}$  gets partitioned into two non-empty subsets  $\mathcal{I}$  and  $\mathcal{J}$  such that the disks corresponding to indices from  $\mathcal{I}$  cover the points  $z_i$  with indices  $i \in I$ , and similarly for  $J$  and  $\mathcal{J}$ .

Now, we put

$$L_n f = \sum_{i: z_i \in D(w_n, \rho)} [\alpha_i f(z_i) + \beta_i f'(z_i)]$$

and note that  $L_n f = (L_n \circ T_{w_n}^{-1})(T_{w_n} f) = \tilde{L}_n(T_{w_n} f)$ , where  $\tilde{L}_n$  is some linear functional of the kind (17) in whose definition the points  $z_i - w_n$  with  $z_i \in D(w_n, \rho)$  participate<sup>2</sup>. Observe that

$$\begin{aligned} \mathcal{E} |L^I F|^2 &= \sum_{n \in \mathcal{I}} \mathcal{E} \left| \tilde{L}_n T_{w_n} F \right|^2 + \sum_{n', n'' \in \mathcal{I}, n' \neq n''} \underbrace{\mathcal{E} \left\{ (\tilde{L}_{n'} T_{w_{n'}} F) (\overline{\tilde{L}_{n''} T_{w_{n''}} F}) \right\}}_{=\sigma(n', n'')} \\ &= \sum_{n \in \mathcal{I}} \mathcal{E} \left| \tilde{L}_n F \right|^2 + \sum_{n', n'' \in \mathcal{I}, n' \neq n''} \sigma(n', n''). \end{aligned}$$

By Claim 3.1, for  $n' \neq n''$ , we have

$$\begin{aligned} |\sigma(n', n'')| &\leq C(\rho, k) e^{-\frac{1}{2}(\underline{d}(\rho) - 4\rho)^2} \left[ \mathcal{E} \left| \tilde{L}_{n'} F \right|^2 + \mathcal{E} \left| \tilde{L}_{n''} F \right|^2 \right] \\ &\leq \frac{1}{10k} \left[ \mathcal{E} \left| \tilde{L}_{n'} F \right|^2 + \mathcal{E} \left| \tilde{L}_{n''} F \right|^2 \right] \end{aligned}$$

by our choice of  $\underline{d}(\rho)$ , so

$$\mathcal{E} |L^I F|^2 \geq \frac{1}{2} \sum_{n \in \mathcal{I}} \mathcal{E} \left| \tilde{L}_n F \right|^2.$$

A similar estimate holds for  $L^J$ , whence,

$$\mathcal{E} |L^I F|^2 + \mathcal{E} |L^J F|^2 \geq \frac{1}{2} \sum_n \mathcal{E} \left| \tilde{L}_n F \right|^2.$$

On the other hand,

$$\left| \mathcal{E} \left\{ (L^I F) (\overline{L^J F}) \right\} \right| \leq \sum_{n' \in \mathcal{I}, n'' \in \mathcal{J}} |\sigma(n', n'')|.$$

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<sup>2</sup>More explicitly, if  $Lf = \sum_i [\alpha_i f(z_i) + \beta_i f'(z_i)]$ , then

$$\tilde{L}_n g = \sum_{i: z_i \in D(w_n, \rho)} [(\alpha_i + \beta_i \overline{w_n}) e^{z_i \overline{w_n}} e^{-\frac{1}{2}|w|^2} g(z_i - w_n) + \beta_i e^{z_i \overline{w_n}} e^{-\frac{1}{2}|w|^2} g'(z_i - w_n)].$$

But for  $n' \in \mathcal{I}$ ,  $n'' \in \mathcal{J}$ , we have

$$|w_{n'} - w_{n''}| \geq |z_i - z_j| - 2\rho \geq d - 2\rho,$$

where  $z_i$  is any point in  $D(w_{n'}, \rho)$ , and  $z_j$  is any point in  $D(w_{n''}, \rho)$ . Using Claim 3.1 again, we get

$$|\sigma(n', n'')| \leq C(\rho, k) e^{-\frac{1}{2}(d-6\rho)^2} \left[ \mathcal{E} \left| \tilde{L}_{n'} F \right|^2 + \mathcal{E} \left| \tilde{L}_{n''} F \right|^2 \right],$$

and

$$\sum_{n' \in \mathcal{I}, n'' \in \mathcal{J}} |\sigma(n', n'')| \leq C(\rho, k) e^{-\frac{1}{2}(d-6\rho)^2} \sum_n \mathcal{E} \left| \tilde{L}_n F \right|^2.$$

Recalling that  $\rho \leq \rho(k)$ , we get the claim.  $\square$

Now, we are ready to prove Theorem 1.2, that is, the multiplicative clustering. To simplify our notation, we denote by

$$\varepsilon = \varepsilon(d, k) = C(k) e^{-\frac{1}{2}(d-\Delta)^2}$$

the small factor on the right-hand side of (22). Recall that  $d$  is a shortcut for  $d(Z_I, Z_J)$  and that  $d \geq 2\Delta$ . In what follows, we assume that  $\Delta$  is so large that  $\varepsilon < \frac{1}{2}$ . We will show that

$$(23) \quad \left[ \frac{1-\varepsilon}{1+\varepsilon} \right]^{2k} \rho(Z_I) \rho(Z_J) \leq \rho(Z) \leq \left[ \frac{1+\varepsilon}{1-\varepsilon} \right]^{2k} \rho(Z_I) \rho(Z_J),$$

which yields Theorem 1.2.

To prove (23), we repeat the argument already used in the proof of Theorem 1.1, and interpret inequality (22) from the previous claim as a two-sided estimate for the covariance matrix  $\Gamma = \Gamma_F(z_1, \dots, z_k)$  of the family of  $2k$  Gaussian random variables  $(F(z_i), F'(z_i))$  ( $1 \leq i \leq k$ ). Consider also the covariance matrices  $\Gamma_I$  and  $\Gamma_J$  of the subfamilies with  $i \in I$  and  $i \in J$  respectively. Then estimate (22) can be rewritten in the following form

$$(24) \quad (1 - \varepsilon) \Gamma_{I,J} \leq \Gamma \leq (1 + \varepsilon) \Gamma_{I,J},$$

where

$$\Gamma_{I,J} = \begin{pmatrix} \Gamma_I & 0 \\ 0 & \Gamma_J \end{pmatrix}$$

is the covariance matrix corresponding to two independent copies  $F_I$  and  $F_J$  of the G.E.F.  $F$ . Inequalities (24) yield

$$(1 - \varepsilon)^{2k} \det \Gamma_{I,J} \leq \det \Gamma \leq (1 + \varepsilon)^{2k} \det \Gamma_{I,J}.$$

We again use the Kac-Rice-Hammersley formula (15):

$$\rho(z_1, \dots, z_k) = \frac{1}{\pi^{2k} \det \Gamma} \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} \langle \Gamma^{-1} \eta', \eta' \rangle} dA(\eta_1) \dots dA(\eta_k),$$

where  $\eta' = (0, \eta_1, \dots, 0, \eta_k)^T$  is a vector in  $\mathbb{C}^{2k}$ . Note now that

$$(1 + \varepsilon)^{-2k} (\det \Gamma_I)^{-1} (\det \Gamma_J)^{-1} \leq (\det \Gamma)^{-1} \leq (1 - \varepsilon)^{-2k} (\det \Gamma_I)^{-1} (\det \Gamma_J)^{-1},$$

and that  $\pi^{2k} = \pi^{2i} \cdot \pi^{2j}$ , where  $i = |I|$  and  $j = |J|$ . To compare the integrals, observe that

$$\begin{aligned}
& \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} \langle \Gamma^{-1} \eta', \eta' \rangle} dA(\eta_1) \dots dA(\eta_k) \\
& \geq \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} (1-\varepsilon)^{-1} \langle \Gamma_{I,J}^{-1} \eta', \eta' \rangle} dA(\eta_1) \dots dA(\eta_k) \\
& = (1-\varepsilon)^{2k} \int_{\mathbb{C}^k} |\eta_1|^2 \dots |\eta_k|^2 e^{-\frac{1}{2} \langle \Gamma_{I,J}^{-1} \eta', \eta' \rangle} dA(\eta_1) \dots dA(\eta_k) \\
& = (1-\varepsilon)^{2k} \int_{\mathbb{C}^i} |\eta_1|^2 \dots |\eta_i|^2 e^{-\frac{1}{2} \langle \Gamma_I^{-1} \eta'_I, \eta'_I \rangle} dA(\eta_1) \dots dA(\eta_i) \\
& \quad \times \int_{\mathbb{C}^j} |\eta_1|^2 \dots |\eta_j|^2 e^{-\frac{1}{2} \langle \Gamma_J^{-1} \eta'_J, \eta'_J \rangle} dA(\eta_1) \dots dA(\eta_j),
\end{aligned}$$

where  $\eta' = (0, \eta_1, \dots, 0, \eta_k)^T$ ,  $\eta'_I = (0, \eta_1, \dots, 0, \eta_i)^T$ , and  $\eta'_J = (0, \eta_1, \dots, 0, \eta_j)^T$ . Similarly, we get an upper bound of the same kind with the factor  $(1+\varepsilon)^{2k}$ . This proves (23) and hence proves Theorem 1.2.  $\square$

#### 4. CENTRAL LIMIT THEOREM VIA CLUSTERING AND CUMULANTS. PROOF OF THEOREM 1.5

The proof of Theorem 1.5 is based on the estimates of cumulants  $s_k = s_k(h)$  of the linear statistics  $n(h) = \sum_{a \in \mathcal{Z}} h(a)$  defined by the formal expansion

$$\log \mathcal{E} \{ e^{tn(h)} \} = \sum_{k \geq 1} \frac{s_k(h)}{k!} t^k.$$

We will show that for each bounded measurable function  $h$  with compact support  $\text{spt}(h)$  and for each  $k \geq 2$ ,

$$|s_k(h)| \leq C_k A(\text{spt}(h)) \|h\|_\infty^k.$$

The key point of this estimate is that applying it to the scaled statistics  $n(R; h)$  we get the factor  $R^2$  on the right-hand side. Together with the growth of the standard deviation  $\sigma(R; h)$  as a positive power of  $R$ , this will yield the asymptotic normality of  $n(R; h)$  when  $R \rightarrow \infty$ .

First, we introduce some notation and recall well-known facts about the moments and cumulants. We denote by  $\Pi(k, j)$  the collection of all unordered partitions of the set  $\{1, 2, \dots, k\}$  into  $j$  disjoint nonempty blocks, and by  $\Pi(k)$  the collection of all unordered partitions of the set  $\{1, 2, \dots, k\}$  into disjoint nonempty blocks. For  $\pi \in \Pi(k)$ , we denote the blocks by  $\{\pi_1, \dots, \pi_j\}$  with an arbitrarily chosen enumeration, and denote the lengths of the blocks by  $p_t = |\pi_t|$ ,  $1 \leq t \leq j$ . The number of blocks in the partition  $\pi$  will be denoted by  $|\pi|$ .

The moments  $m_k$  and the cumulants  $s_k$  are related to each other by classical formulas [15, Chapter II, § 12]:

$$(25) \quad s_k = \sum_{\ell=1}^k (-1)^{\ell-1} (\ell-1)! \sum_{\pi \in \Pi(k, \ell)} m_{p_1} \dots m_{p_\ell}.$$

The inverse formula has the form

$$(26) \quad m_k = \sum_{\pi \in \Pi(k)} s_{p_1} \dots s_{p_t}.$$

Next, we recall the definition of so called *truncated* (or Ursell)  $k$ -point functions  $\rho_k^T$ . The truncated  $k$ -point functions play the same rôle for the cumulants as the usual  $k$ -point functions for the moments. Their definition is suggested by (25). They are symmetric functions

$$\rho_k^T : \{Z = (z_1, \dots, z_k) \in \mathbb{C}^k : z_i \neq z_j, \text{ for } i \neq j\} \rightarrow \mathbb{R}$$

defined by the formula

$$(27) \quad \rho_k^T(Z) = \sum_{\ell=1}^k (-1)^{\ell-1} (\ell-1)! \sum_{\pi \in \Pi(k, \ell)} \rho_{p_1}(Z_{\pi_1}) \dots \rho_{p_\ell}(Z_{\pi_\ell}).$$

In particular,  $\rho_1^T = \rho_1$ ,

$$(28) \quad \rho_2^T(z_1, z_2) = \rho_2(z_1, z_2) - \rho_1(z_1)\rho_1(z_2),$$

then

$$\begin{aligned} \rho_3^T(z_1, z_2, z_3) &= \rho_3(z_1, z_2, z_3) \\ &\quad - (\rho_2(z_1, z_2)\rho_1(z_3) + \rho_2(z_1, z_3)\rho_1(z_2) + \rho_2(z_2, z_3)\rho_1(z_1)) \\ &\quad + 2\rho_1(z_1)\rho_1(z_2)\rho_1(z_3), \end{aligned}$$

and so on ... The inversions to (27) look the same as in (26):

$$(29) \quad \rho_k(Z) = \sum_{\pi \in \Pi(k)} \rho_{p_1}^T(Z_{\pi_1}) \dots \rho_{p_t}^T(Z_{\pi_t}).$$

See e.g. [8, Appendix A.7] for the proof of the equivalence of the formulas (27) and (29) based on generating functions; another short and direct proof of this equivalence can be found in [1, § 2].

Asymptotic factorization of  $k$ -point correlation functions expressed in (8) yields fast asymptotic decay of truncated  $k$ -point functions when the diameter of the configuration gets large.

**Claim 4.1.** *Suppose that the  $k$ -point functions  $\rho_k$  of a point process  $\mathcal{Z}$  are clustering, i.e., satisfy (8) with a fast decreasing function  $\varphi$ , and are bounded. Then,*

$$(30) \quad |\rho_k^T(Z)| \leq C_k \tilde{\varphi}(c_k \text{diam}(Z)),$$

where  $\tilde{\varphi} = \min(1, \varphi)$  and  $\text{diam}(Z) = \max_{1 \leq i < j \leq k} |z_i - z_j|$  is the diameter of the configuration  $Z$ .

*Proof of Claim 4.1:* We show that for each  $k \geq 2$  and for each partition of the set of indices  $\{1, 2, \dots, k\} = I \sqcup J$ ,

$$(31) \quad |\rho_k^T(Z)| \leq C_k \tilde{\varphi}(c_k d(Z_I, Z_J)).$$

Since for each configuration  $Z$ , there exists a partition  $\{1, 2, \dots, k\} = I \sqcup J$  with  $d(Z_I, Z_J) \geq c_k \text{diam}(Z)$ , estimate (31) yields (30).

To prove (31), we use induction on  $k$ . For  $k = 2$ , estimate (31) follows from equation (28), clustering of  $\rho_2$ , and boundedness of  $\rho_1$  and  $\rho_2$ . Now, suppose that  $k \geq 3$ . We fix a partition  $\{1, 2, \dots, k\} = I \sqcup J$  with  $i = |I|$  and  $j = |J|$ . We say that a partition



$\pi \in \Pi(k)$  *refines* the partition  $I \sqcup J$  if each set  $\pi_s$  is a subset of either  $I$ , or  $J$ . Otherwise, we say that the partition  $\pi$  *mixes*  $I \sqcup J$ . By the inversion formula (29), we have

$$\rho_k^T(Z) = \rho_k(Z) - \sum_{t=|\pi| \geq 2} \rho_{p_1}^T(Z_{\pi_1}) \dots \rho_{p_t}^T(Z_{\pi_t}),$$

and

$$\rho_i(Z_I) \rho_j(Z_J) = \sum_{\pi \text{ refines } I \sqcup J, t=|\pi| \geq 2} \rho_{p_1}^T(Z_{\pi_1}) \dots \rho_{p_t}^T(Z_{\pi_t}).$$

Therefore,

$$\rho_k^T(Z) = \rho_k(Z) - \rho_i(Z_I) \rho_j(Z_J) + \sum_{\pi \text{ mixes } I \sqcup J, t=|\pi| \geq 2} \rho_{p_1}^T(Z_{\pi_1}) \dots \rho_{p_t}^T(Z_{\pi_t}).$$

By the induction assumption, each term in the sum on the RHS contains at least one factor which is bounded by the right-hand side of (31), while the other factors are bounded by constants.  $\square$

**Claim 4.2.** *Suppose that the truncated  $k$ -point functions  $\rho_k^T$  of a point process  $\mathcal{Z}$  satisfy (30) with a fast decreasing function  $\varphi$ . Then, for each  $k \geq 2$ ,*

$$(32) \quad \sup_{z_k \in \mathbb{C}} \int_{\mathbb{C}^{k-1}} |\rho_k^T(Z)| \, dA(z_1) \dots dA(z_{k-1}) < \infty.$$

*Proof of Claim 4.2:* We fix a point  $z_k \in \mathbb{C}$ , and set  $Z = (Z', z_k)$ , where  $Z' \in \mathbb{C}^{k-1}$ . Then we split  $\mathbb{C}^{k-1}$  into disjoint sets  $G_\ell$ :

$$G_0 = \{Z' : \text{diam}(Z) \leq 1\}, \text{ and } G_\ell = \{Z' : 2^{\ell-1} < \text{diam}(Z) \leq 2^\ell\} \text{ for } \ell \geq 1.$$

Applying estimate (30), we get  $|\rho_k^T(Z)| \leq C_k \varphi(c_k 2^{\ell-1})$  provided that  $Z' \in G_\ell$  with  $\ell \geq 1$ . Then the integral of the absolute value of  $\rho_k^T$  over  $G_\ell$  does not exceed

$$C_k 2^{2\ell(k-1)} \tilde{\varphi}(c_k 2^{\ell-1}).$$

Hence, the integral on the left-hand side of (32) does not exceed

$$C_k + C_k \sum_{\ell \geq 1} 2^{2\ell(k-1)} \tilde{\varphi}(c_k 2^{\ell-1}) \lesssim C_k$$

(in the last estimate we use that the function  $\varphi$  decreases faster than any power).  $\square$

The computation of the  $k$ -th cumulant  $s_k(h)$  of the linear statistics  $n(h)$  requires knowledge of  $j$ -point truncated functions with all  $j \leq k$ . For bounded compactly supported measurable functions  $h_1, \dots, h_k$  on  $\mathbb{C}$ , we put

$$\langle h_1, \dots, h_k \rangle^T \stackrel{\text{def}}{=} \int_{\mathbb{C}^k} h_1(z_1) \dots h_k(z_k) \rho^T(z_1, \dots, z_k) \, dA(z_1) \dots dA(z_k).$$

Given a set  $P = \{p_1, \dots, p_k\}$  of  $k$  positive integers, we define

$$s(h, P) = \langle h^{p_1}, \dots, h^{p_k} \rangle^T.$$

Given a partition  $\gamma$ , we denote by  $P_\gamma = \{|\gamma_1|, \dots, |\gamma_j|\}$  the set whose elements are the lengths of the blocks of  $\gamma$ .

**Claim 4.3.** *We have*

$$(33) \quad s_k(h) = \sum_{\gamma \in \Pi(k)} s(h, P_\gamma).$$

Probably, these relations are well known to those who worked with  $k$ -point functions of point processes. For this reason, we relegate the proof of Claim 4.3 to the appendix.

Now, we easily estimate the cumulants of the linear statistics  $n(h)$ :

**Claim 4.4.** *For each bounded compactly supported function  $h$ , we have*

$$|s_k(h)| \leq C_k \|h\|_\infty^k A(\text{spt}(h)).$$

*Proof of Claim 4.4:* By equation (33), it suffices to estimate from above  $|\langle h^{p_1}, \dots, h^{p_j} \rangle^T|$  where  $p_1, \dots, p_j$  are positive integers such that  $\sum_j p_j = k$ .

Let  $G = \text{spt}(h)$ . Then

$$\begin{aligned} |\langle h^{p_1}, \dots, h^{p_j} \rangle^T| &\leq \int_{\mathbb{C}^j} |h(z_1)|^{p_1} \dots |h(z_j)|^{p_j} |\rho^T(z_1, \dots, z_j)| dA(z_1) \dots dA(z_j) \\ &\leq \|h\|_\infty^k \int_{\mathbb{C}^j} \mathbb{1}_G(z_1) \dots \mathbb{1}_G(z_j) |\rho^T(z_1, \dots, z_j)| dA(z_1) \dots dA(z_j) \\ &\leq \|h\|_\infty^k \int_{\mathbb{C}} \mathbb{1}_G(z_j) dA(z_j) \int_{\mathbb{C}^{j-1}} |\rho^T(z_1, \dots, z_{j-1})| dA(z_1) \dots dA(z_{j-1}) \\ &\stackrel{(32)}{\leq} C_k \|h\|_\infty^k \int_{\mathbb{C}} \mathbb{1}_G(z_j) dA(z_j) = C_k \|h\|_\infty^k A(G), \end{aligned}$$

proving the claim.  $\square$

*Proof of Theorem 1.5:* Denoting by  $s_k(R; h)$  the cumulants of  $n(R; h)$ , and by  $s_k^*(R, h)$  the cumulants of the normalized random variables  $n^*(R; h)$ , we have  $s_1^* = 0$ ,  $s_2^* = 1$ , and for  $k \geq 3$ ,

$$s_k^*(R; h) = \frac{s_k(R; h)}{\sigma(R; h)^k}.$$

By the estimate from the previous claim, we have

$$|s_k(R; h)| \leq C(k, h) R^2.$$

Recalling that  $\sigma(R; h)$  grows as a power of  $R$ , we see that for large enough  $k$ 's,

$$\lim_{R \rightarrow \infty} s_k^*(R; h) = 0.$$

By a version of the classical theorem of Marcinkiewicz (see [6, 17]), this suffices to conclude that the random variables  $n^*(R; h)$  converge in distribution to the Gaussian law when  $R \rightarrow \infty$ . This finishes off the proof of Theorem 1.5.  $\square$

## 5. LOWER BOUND FOR THE VARIANCE. PROOF OF LEMMAS 1.6 AND 1.7

First, we prove Lemma 1.7.

*Proof of Lemma 1.7:* It suffices to prove the lemma for  $R = 1$ . The general case readily follows by scaling

$$x \mapsto Rx, \quad h \mapsto h(R^{-1} \cdot), \quad \xi \mapsto R^{-1} \xi, \quad \hat{h} \mapsto R^2 \hat{h}(R \cdot).$$

Recall that  $\rho(z_1, z_2) = r(z_1 - z_2)$ , and let  $\kappa(z) \stackrel{\text{def}}{=} r(z) - 1$ . By the assumptions of the lemma, this is an  $L^1(\mathbb{R}^2)$ -function. We have

$$\begin{aligned} \sigma(1; h)^2 &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} h(z_1)h(z_2) [\rho(z_1, z_2) + \delta(z_1 - z_2) - 1] \, dA(z_1)dA(z_2) \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} h(z_1)h(z_2) [\kappa(z_1 - z_2) + \delta(z_1 - z_2)] \, dA(z_1)dA(z_2) \\ &= \int_{\mathbb{R}^2} |\widehat{h}(\xi)|^2 [1 + \widehat{\kappa}(\xi)] \, dA(\xi) \end{aligned}$$

(in the last line we used Parseval's formula). By the Riemann-Lebesgue lemma,  $\widehat{\kappa}(\xi) \rightarrow 0$  when  $|\xi| \rightarrow \infty$ . Thus, choosing a sufficiently big constant  $C$  so that  $|\widehat{\kappa}(\xi)| \leq \frac{1}{2}$  for  $|\xi| \geq C$ , we get

$$\sigma(1; h)^2 \geq \frac{1}{2} \int_{|\xi| \geq C} |\widehat{h}(\xi)|^2 \, dA(\xi),$$

proving the lemma.  $\square$

**Remark 5.1** (cf. Martin-Yalçin [9]). The computation above gives us

$$\begin{aligned} \sigma(R; h)^2 &= R^2 \int_{\mathbb{R}^2} |\widehat{h}(\xi)|^2 [1 + \widehat{\kappa}(R^{-1}\xi)] \, dA(\xi) \\ &= [1 + \widehat{\kappa}(0) + o(1)] \|h\|_{L^2(\mathbb{R}^2)}^2 R^2, \end{aligned}$$

as  $R \rightarrow \infty$ . We conclude that translation-invariant point processes with  $r - 1 \in L^1(\mathbb{R}^2)$  can be divided into two groups: the processes with the Poissonian behaviour of the variance of linear statistics, when  $\widehat{\kappa}(0) \neq -1$ , i.e.,

$$\int_{\mathbb{R}^2} [r(x) - 1] \, dA(x) \neq -1,$$

and the processes with non-Poissonian behaviour of the variance of linear statistics, when  $\widehat{\kappa}(0) = -1$ , i.e.,

$$(34) \quad \int_{\mathbb{R}^2} [r(x) - 1] \, dA(x) = -1.$$

Sometimes, the latter processes are called “superhomogeneous processes”. Lemma 1.6 proven below is trivial for the processes from the first group. On the other hand, though condition (34) looks like a degeneration, many interesting translation-invariant processes including our main hero, the zero point process  $\mathcal{Z}_F$ , and particle processes based on Coulomb interaction that occur in statistical mechanics belong to the second group.

**Remark 5.2.** For the random zero process  $\mathcal{Z}_F$ , the computation started above can be continued. This way, we arrive at an explicit formula for the variance [10] which yields that

$$\sigma(R; h)^2 \simeq R^{-2} \int_{|\xi| \leq R} |\widehat{h}(\xi)|^2 |\xi|^4 \, dA(\xi) + R^2 \int_{|\xi| \geq R} |\widehat{h}(\xi)|^2 \, dA(\xi).$$

Now, we prove the remaining Lemma 1.6.

*Proof of Lemma 1.6:* Fix a multiplier  $m \in C_0^\infty(\mathbb{R}^2)$  with the following properties:  $0 \leq m \leq 1$  everywhere,  $m(\xi) = 0$  for  $|\xi| \geq 2$ , and  $m(\xi) = 1$  for  $|\xi| \leq 1$ , and set  $m_R(\xi) = m(R^{-1}\xi)$ . We use the function  $m_R$  to cut high-frequency oscillations in the

spectrum of the indicator function  $\mathbb{1}_E$ . Denote by  $\varphi_{E,R}$  the inverse Fourier transform of the product  $\widehat{\mathbb{1}_E} \cdot m_R$ . Since  $\nabla \widehat{m_R}(x) = R^3(\nabla \widehat{m})(Rx)$ , we have  $\|\nabla \widehat{m_R}\|_{L^1} = R\|\nabla \widehat{m}\|_{L^1}$ , whence

$$(35) \quad |\nabla \varphi_{E,R}| \lesssim R.$$

By Lemma 1.7 we have

$$\sigma(R, \mathbb{1}_E)^2 \gtrsim R^2 \int_{|\xi| \geq R'} |\widehat{\mathbb{1}_E}|^2 \geq R^2 \int_{\mathbb{R}^2} |\widehat{\mathbb{1}_E}|^2 |1 - m_{R'}|^2 = R^2 \int_{\mathbb{R}^2} |\mathbb{1}_E - \varphi_{E,R'}|^2,$$

with  $R' = CR$ , where  $C$  is a constant on the right-hand side of estimate (10) in the statement of Lemma 1.7. We estimate the integral on right-hand side twice. The first bound works when  $A(\{\varphi_{E,R'} \geq \frac{1}{2}\}) \geq \frac{1}{2}A(E)$ :

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbb{1}_E - \varphi_{E,R'}|^2 &\geq \int_{\{\varphi_{E,R'} \leq \frac{1}{2}\}} |\mathbb{1}_E - \varphi_{E,R'}|^2 \\ &\geq \frac{1}{4} \int_{\{\varphi_{E,R'} \leq \frac{1}{2}\}} 1 \stackrel{(35)}{\gtrsim} \frac{1}{R} \int_{\{\varphi_{E,R'} \leq \frac{1}{2}\}} |\nabla \varphi_{E,R'}| \\ &= \frac{1}{R} \int_0^{1/2} \text{Length}(\{\varphi_{E,R'} = t\}) dt \quad (\text{coarea formula}) \\ &\geq \frac{1}{R} \int_0^{1/2} \sqrt{4\pi A(\{\varphi_{E,R'} \geq t\})} dt \quad (\text{isoperimetric inequality}) \\ &\gtrsim \frac{1}{R} \sqrt{A(\{\varphi_{E,R'} \geq \frac{1}{2}\})} \geq \frac{1}{R\sqrt{2}} \sqrt{A(E)}. \end{aligned}$$

On the other hand, if  $A(\{\varphi_{E,R'} \geq \frac{1}{2}\}) \leq \frac{1}{2}A(E)$ , we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbb{1}_E - \varphi_{E,R'}|^2 &\geq \int_{\{\varphi_{E,R'} \leq \frac{1}{2}\}} |\mathbb{1}_E - \varphi_{E,R'}|^2 \\ &\geq \frac{1}{4} \int_{\{\varphi_{E,R'} \leq \frac{1}{2}\}} \mathbb{1}_E \gtrsim A(E) - A(\{\varphi_{E,R'} \geq \frac{1}{2}\}) \geq \frac{1}{2}A(E). \end{aligned}$$

In both cases,

$$R^2 \int_{|\xi| \geq 2R} |\widehat{\mathbb{1}_E}|^2 \gtrsim \min \left\{ A(E)R^2, \sqrt{A(E)}R \right\},$$

proving the lemma.  $\square$

#### APPENDIX: PROOF OF CLAIM 4.3

First, we prove relations analogous to (33) that express the  $k$ -th moment  $m_k(h)$  of the linear statistics  $n(h)$  in terms of  $j$ -point functions  $\rho_j$  with  $j \leq k$ . For bounded compactly supported functions  $h_1, \dots, h_k$  on  $\mathbb{C}$ , we put

$$\langle h_1, \dots, h_k \rangle \stackrel{\text{def}}{=} \int_{\mathbb{C}^k} h_1(z_1) \dots h_k(z_k) \rho(z_1, \dots, z_k) dA(z_1) \dots dA(z_k).$$

Given a set  $P = \{p_1, \dots, p_k\}$  of  $k$  positive integers, we define

$$m(h, P) = \langle h^{p_1}, \dots, h^{p_k} \rangle.$$

As above, given a partition  $\gamma$ , we denote by  $P_\gamma = \{|\gamma_1|, \dots, |\gamma_j|\}$  the set whose elements are the lengths of the blocks of  $\gamma$ . Then

$$(36) \quad m_k(h) = \sum_{\gamma \in \Pi(k)} m(h, P_\gamma).$$

To prove this relation, we denote by  $(a_1, \dots, a_k)$  an ordered finite sequence of length  $k$  of points in  $\mathcal{Z}$ , possibly with repetitions. Then

$$m_k(h) = \mathcal{E} \left\{ \sum_{(a_1, \dots, a_k)} h(a_1) \dots h(a_k) \right\}$$

where the sum on the right-hand side is taken over all finite sequences  $(a_1, \dots, a_k)$ . We fix a partition  $\gamma \in \Pi(k)$  and say that the sequence  $(a_1, \dots, a_k)$  is *subordinated* to  $\gamma$  if the following condition holds:  $a_s = a_t$  for  $s \neq t$  if and only if the indices  $s$  and  $t$  belong to the same block of the partition  $\gamma$ . In this case, we'll write  $(a_1, \dots, a_k) \triangleleft \gamma$ . Clearly, each finite sequence of length  $k$  is subordinated to one and only one partition  $\gamma \in \Pi(k)$ .

Given a partition  $\gamma \in \Pi(k, j)$ , we fix an arbitrary enumeration  $\gamma_1, \dots, \gamma_j$  of its blocks. Then we have

$$\sum_{(a_1, \dots, a_k) \triangleleft \gamma} h(a_1) \dots h(a_k) = \sum_{\substack{(b_1, \dots, b_j) \\ b_s \neq b_t \text{ for } s \neq t}} h^{|\gamma_1|}(b_1) \dots h^{|\gamma_j|}(b_j),$$

and then

$$\mathcal{E} \left\{ \sum_{(a_1, \dots, a_k)} h(a_1) \dots h(a_k) \right\} = \sum_{j=1}^k \sum_{\gamma \in \Pi(k, j)} \mathcal{E} \left\{ \sum_{\substack{(b_1, \dots, b_j) \\ b_s \neq b_t \text{ for } s \neq t}} h^{|\gamma_1|}(b_1) \dots h^{|\gamma_j|}(b_j) \right\}.$$

Since

$$\mathcal{E} \left\{ \sum_{\substack{(b_1, \dots, b_j) \\ b_s \neq b_t \text{ for } s \neq t}} h_1(b_1) \dots h_j(b_j) \right\} = \langle h_1, \dots, h_j \rangle$$

(see for instance, [5, Section 1.2]), we complete the proof of relations (36).  $\square$

Now, we turn to the proof of relations (33). Given a partition  $\gamma \in \Pi(k, j)$  we fix an arbitrary enumeration  $\gamma_1, \dots, \gamma_j$  of its blocks. Then according to the definition (27) of truncated functions  $\rho^T$ , we have

$$(37) \quad s(h, P_\gamma) = \sum_{\ell=1}^j (-1)^{\ell-1} (\ell-1)! \\ \times \sum_{\pi \in \Pi(j, \ell)} \int_{\mathbb{C}^j} h^{|\gamma_1|}(z_1) \dots h^{|\gamma_j|}(z_j) \rho(Z_{\pi_1}) \dots \rho(Z_{\pi_\ell}) dA(z_1) \dots dA(z_j)$$

where the partition  $\pi$  splits the set of blocks  $\{\gamma_1, \dots, \gamma_j\}$  into  $\ell$  'super-blocks'  $\Pi^t$ ,  $1 \leq t \leq \ell$ , which we also enumerate arbitrarily. By  $p_t$ ,  $1 \leq t \leq \ell$ , we denote the number of blocks of  $\gamma$  that are included into the super-block  $\Pi^t$ . By  $P^t = P_{\gamma, \pi}^t = \{|\gamma_{\alpha_1}|, \dots, |\gamma_{\alpha_{p_t}}|\}$  we denote the set of the lengths of the blocks  $\gamma_{\alpha_1}, \dots, \gamma_{\alpha_{p_t}}$  that are included into the super-block  $\Pi^t$ , this is the set of  $p_t$  positive integers. For instance, let  $k = 8$ ,  $j = 3$ ,

$\ell = 2$ , let  $\gamma_1 = \{1, 2, 3\}$ ,  $\gamma_2 = \{4, 5, 6\}$ ,  $\gamma_3 = \{7, 8\}$ , and let  $\pi_1 = \{1, 2\}$ ,  $\pi_2 = \{3\}$ . Then  $\Pi^1 = \{\gamma_1, \gamma_2\}$ ,  $\Pi^2 = \{\gamma_3\}$ ,  $p_1 = 2$ ,  $p_2 = 1$ , and  $P^1 = \{3, 3\}$ ,  $P^2 = \{2\}$ .

We factor the integrals on the right-hand side of (37)

$$\int_{\mathbb{C}^j} h^{|\gamma_1|}(z_1) \dots h^{|\gamma_j|}(z_j) \rho(Z_{\pi_1}) \dots \rho(Z_{\pi_\ell}) dA(z_1) \dots dA(z_j) = \prod_{t=1}^{\ell} m(h, P_{\gamma, \pi}^t).$$

Then the right-hand side of (33) equals

$$(38) \quad \sum_{j=1}^k \sum_{\gamma \in \Pi(k, j)} \sum_{\ell=1}^j (-1)^{\ell-1} (\ell-1)! \sum_{\pi \in \Pi(j, \ell)} \prod_{t=1}^{\ell} m(h, P_{\gamma, \pi}^t).$$

We say that a partition  $\gamma \in \Pi(k, j)$  *refines* partition  $\sigma \in \Pi(k, \ell)$ ,  $\ell \leq j$ , if each block of  $\gamma$  is contained in one of the blocks of  $\sigma$ . In this case, we write  $\gamma \prec \sigma$ . Given  $1 \leq \ell \leq j \leq k$ , there is one-to-one correspondence between all possible pairs of partitions  $(\gamma, \pi)$  with  $\gamma \in \Pi(k, j)$  and  $\pi \in \Pi(j, \ell)$  splitting the set  $\{\gamma_1, \dots, \gamma_j\}$  of blocks of  $\gamma$  into  $\ell$  ‘super-blocks’, and all possible pairs of partitions  $(\gamma, \sigma)$  with  $\gamma \in \Pi(k, j)$  and  $\sigma \in \Pi(k, \ell)$  such that  $\gamma \prec \sigma$ . We will use the enumeration  $\sigma_1, \dots, \sigma_\ell$  of blocks of  $\sigma$  induced by the enumeration of blocks of  $\pi$ . For instance, in the example considered above,  $\sigma \in \Pi(8, 2)$ ,  $\sigma_1 = \{1, 2, 3, 4, 5, 6\}$  and  $\sigma_2 = \{7, 8\}$ .

We denote by  $P_{\gamma, \sigma}^t = P_{\gamma, \pi}^t$ ,  $1 \leq t \leq \ell$ , the set of lengths of blocks of  $\gamma$  that are contained in the block  $\sigma_t$  of  $\sigma$ . Then changing the order of summations in (38), we get

$$\begin{aligned} \sum_{\ell=1}^k (-1)^{\ell-1} (\ell-1)! \sum_{\sigma \in \Pi(k, \ell)} \sum_{j=\ell}^k \sum_{\substack{\gamma \in \Pi(k, j) \\ \gamma \prec \sigma}} \prod_{t=1}^{\ell} m(h, P_{\gamma, \sigma}^t) \\ = \sum_{\ell=1}^k (-1)^{\ell-1} (\ell-1)! \sum_{\sigma \in \Pi(k, \ell)} \sum_{\substack{\gamma \in \Pi(k) \\ \gamma \prec \sigma}} \prod_{t=1}^{\ell} m(h, P_{\gamma, \sigma}^t). \end{aligned}$$

Next, given  $\sigma \in \Pi(k, \ell)$ , we fix an arbitrary enumeration of its blocks, put  $q_t = |\sigma_t|$ ,  $1 \leq t \leq \ell$ , and replace one partition  $\gamma \prec \sigma$  by  $\ell$  partitions  $\gamma^1 \in \Pi(q_1)$ ,  $\dots$ ,  $\gamma^\ell \in \Pi(q_\ell)$  that split the blocks of  $\sigma$  into the corresponding sub-blocks. In the example considered above,  $q_1 = 6$ ,  $q_2 = 2$ , the partition  $\gamma^1$  consists of two blocks  $\{1, 2, 3\}$  and  $\{4, 5, 6\}$ , and the partition  $\gamma^2$  consists of one block.

At last, replacing the sum  $\sum_{\substack{\gamma \in \Pi(k) \\ \gamma \prec \sigma}}$  by the  $\ell$ -tuple sum  $\sum_{\gamma^1 \in \Pi(q_1)} \dots \sum_{\gamma^\ell \in \Pi(q_\ell)}$ , we get

$$\sum_{\substack{\gamma \in \Pi(k) \\ \gamma \prec \sigma}} \prod_{t=1}^{\ell} m(h, P_{\gamma, \sigma}^t) = \prod_{i=1}^{\ell} \sum_{\lambda \in \Pi(q_i)} m(h, P_\lambda) \stackrel{(36)}{=} m_{q_1}(h) \dots m_{q_\ell}(h).$$

Hence, the right-hand side of (33) equals

$$\sum_{\ell=1}^k (-1)^{\ell-1} (\ell-1)! \sum_{\sigma \in \Pi(k, \ell)} m_{q_1}(h) \dots m_{q_\ell}(h) \stackrel{(25)}{=} s_k(h),$$

proving the claim.  $\square$

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